ver's discussion below, covering substantially the ground of a paper presented by him at the summer meeting of the Mathematical Association of America in September, 1918, should be suggestive and stimulating, and will, it is hoped, give rise to further expressions of opinion.

TRIGONOMETRIC FUNCTIONS—OF WHAT?[1]

BY W. B. CARVER, Cornell University.

The first idea which our students get of a trigonometric function—say sin x —is that the argument x is an *angle,* a geometric entity. According to this conception, the sine of a right angle is 1 whether one thinks of it as an angle of 90° or of $\pi/2$ radians; while sin 2 has different values according as the 2 means 2 radians, 2 degrees, or 2 right angles.[2] Later the student needs the notion of the number sin x as a function of the *number* x, a functional relation which is not dependent upon any sort of geometric ideas or units of geometric measurement. This second point of view is needed not only by those who specialize in mathematics, but also by the large class of students who go no further than a first course in the calculus, and whose purpose may be entirely utilitarian.

It is the writer's conviction that, certainly in our text-books, and possibly in our teaching, we are not doing as much as we might to help the student across from the one point of view to the other.

In our courses in trigonometry the first point of view must prevail: but the way may be prepared for the second by insisting upon familiarity with the circular measurement of angles. The radian unit should be introduced early and *used frequently* throughout the course. The tables of functions should have a column giving the angle in radians adjacent to the column reading degrees and minutes.[3] In the problems for solution in both right and oblique triangles, the given angles should, in at least a few cases, be expressed in radians. The relation of the number π to this method of measuring angles should be made clear. That there is confusion on this point is indicated by such questions as "Why does π mean 3.1416 in one place, and in another place 180°?"

Analytic geometry should bring us nearer to the idea of a trigonometric function of a number. Should a student be permitted to draw the curve represented by the equation $y = \sin x$ with the wave-length and amplitude in any ratio that pleases him? If so, may he plot $4x^2 + y^2 = 1$ as a circle, or $x^2 + y^2 = 1$ as an ellipse? In drawing such trigonometric curves, should we not insist that the units of length on the x and y axes shall be the same, and that sin x means the sine of an angle of x radians? Queerly enough, in polar coördinates the trouble arises when we do *not* have trigonometric functions of θ rather than

[1] Read before the Mathematical Association of America, September 6, 1918.

[2] Many of the text-books state explicitly the convention that the radian is to be assumed when no other unit is indicated.

[3] Such tables are surprisingly scarce; as are also protractors reading radians and decimal parts of radians.

when we do. Why does the curve $\rho = \theta^2$ raise the question of the unit of angle measurement while the curve $\rho = \sin\theta$ does not? Is the θ of our polar coördinates a number or an angle? The plotting of the curve $\sin\rho = \theta$ presents the difficulty in even more acute form. The best way out of all these difficulties would seem to be to emphasize the fact that ρ and θ are numbers; and that in polar coördinates ρ is represented graphically by ρ units (any convenient unit) of length, while θ is represented by an angle of θ *radians*.

It is in the course in calculus that the student must be brought to a full realization of the significance of $\sin x$ as a function of a number. In deducing the formula $d\sin x/dx = \cos x$, the idea of x as an angle is still in evidence; but it should be emphasized that the validity of the formula depends upon the fact that x means the number of *radians* in the angle. One good method of impressing this point is *first* to deduce a formula assuming that x means an angle of x degrees, so that the student may see clearly the advantage of the use of the radian unit. Being thus committed to this unit of angle measurement, it will follow later that in the formula

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C,$$

$\sin^{-1} x$ means the number of radians in the angle[1] whose sine is x. How many of our students have any reason for a choice between $\pi/4$ and $45°$ as a value for the integral $\int_0^1 \sqrt{1-x^2}dx$, other than the fact that the latter value does not seem to give the "right answer" for the area of the quarter circle?

When we have expanded $\sin x$ in a power series, convergent (happily) for all values of x, we have finally the basis for a definition of the function which is independent of geometric notions. An analogy may be helpful to the student at this point. The geometric notions of the area and length of side of a square may be used to exhibit the relation between the numbers x and $\sqrt{x}$; but nevertheless this number relation is independent of such geometric considerations. And the student of fairly keen mathematical insight will be interested to see that he now has a relation between the numbers x and $\sin x$ which is similarly independent of geometric considerations.

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REVIEWS.

MATHEMATICS FOR FRESHMEN.

1. *Introduction to the Elementary Functions.* By R. B. McCLENON with the editorial coöperation of W. J. Rusk. Boston, Ginn, 1918. 8vo. 9 + 244 pp. Price $1.80.

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[1] It is also important, of course, that $\sin^{-1} x$ should have been so defined as to make it a single-valued and continuous function.